

Admissible sheaves on \mathbb{P}^3

Marcos Jardim
 IMECC - UNICAMP
 Departamento de Matemática
 Caixa Postal 6065
 13083-970 Campinas-SP, Brazil

February 1, 2008

Abstract

Admissible locally-free sheaves on \mathbb{P}^3 , also known in the literature as mathematical instanton bundles, arise in twistor theory, and are in 1-1 correspondence with instantons on \mathbb{R}^4 . In this paper, we study admissible sheaves on \mathbb{P}^3 from the algebraic geometric point of view. We discuss examples and compare the admissibility condition with semistability and splitting type.

Contents

1	Monads	3
2	Examples of admissible sheaves	7
3	Semistability of torsion-free admissible sheaves	9
4	Trivial splitting type	12
5	Open problems	15

Introduction

An intense interest on the construction and classification of locally-free sheaves on the 3-dimensional complex projective space started on the late 70's, when twistor theory yielded a 1-1 correspondence between instantons (i.e. anti-self-dual connections of finite L^2 norm) on \mathbb{R}^4 and certain holomorphic vector bundles on \mathbb{P}^3 ; this is the celebrated Penrose-Ward correspondence [7, 10]. This fact was later used by Atiyah, Drinfeld, Hitchin and Manin to construct and classify all instantons [1].

Since then, many authors have studied the so-called *mathematical* (or *complex*) *instanton bundles*, defined in the literature as rank 2 locally-free sheaves E on \mathbb{P}^3 with $c_1(E) = c_3(E) = 0$ and $c_2(E) = c > 0$ satisfying $H^0(\mathbb{P}^3, E(-1)) = H^1(\mathbb{P}^3, E(-2)) = 0$; see [2] for a recent brief survey of this topic. These correspond to $SL(2, \mathbb{C})$ instantons on \mathbb{R}^4 of charge c . The correct generalization for higher rank sheaves is given by Manin and Drinfeld (see [7]), and leads us to the key definition of this paper:

Definition. *An admissible sheaf on \mathbb{P}^3 is a coherent sheaf E satisfying:*

$$H^p(\mathbb{P}^3, E(k)) = 0 \quad \text{for } p \leq 1, \quad p + k \leq -1, \quad \text{and } p \geq 2, \quad p + k \geq 0$$

Admissible locally-free sheaves of rank r and vanishing first Chern class are in 1-1 correspondence with $SL(r, \mathbb{C})$ instantons on \mathbb{R}^4 [7]. In this paper, we study mainly torsion-free admissible sheaves with vanishing first Chern class, which can be regarded as a generalization of instantons. Our focus is on the algebraic geometric properties of such objects, like semistability and splitting type.

The paper is organized as follows. In Section 1 we remark that admissible sheaves are in 1-1 correspondence with certain monads, exploring a few properties and some examples in Section 2. We then discuss how admissibility and semistability compare with one another in Section 3, and conclude with an analysis of the splitting type of torsion-free admissible sheaves with vanishing first Chern class in the last section.

Acknowledgment. Some of the results presented here were obtained in joint work with Igor Frenkel [4]; we thank him for his continued support. We also thank the organizers and participants of the XVIII Brazilian Algebra Meeting.

1 Monads

Let X be a smooth projective variety. A *monad* on X is a sequence V_\bullet of the following form:

$$\mathcal{V}_\bullet : 0 \rightarrow V_{-1} \xrightarrow{\alpha} V_0 \xrightarrow{\beta} V_1 \rightarrow 0 \quad (1)$$

which is exact on the first and last terms. Here, V_k are locally free sheaves on X . The sheaf $E = \ker \beta / \operatorname{Im} \alpha$ is called the cohomology of the monad \mathcal{V}_\bullet , also denoted by $H^1(\mathcal{V}_\bullet)$.

In this paper, we will focus on the so-called *special monads* on \mathbb{P}^3 , which are of the form:

$$0 \rightarrow V \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} W \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\beta} V' \otimes \mathcal{O}_{\mathbb{P}^3}(1) \rightarrow 0 ,$$

where α is injective and β is surjective. The existence of such objects has been completely classified by Floystad in [3]; let $v = \dim V$, $w = \dim W$ and $v' = \dim V'$.

Theorem 1. *There exists a special monad on \mathbb{P}^3 as above if and only if at least one of the following conditions hold:*

- $w \geq 2v' + 2$ and $w \geq v + v'$;
- $w \geq v + v' + 3$.

Monads appeared in a wide variety of contexts within algebraic geometry, like the construction of locally free sheaves on complex projective spaces, the study of curves in \mathbb{P}^3 and surfaces in \mathbb{P}^4 . In this section, we will see how they are related to admissible sheaves on \mathbb{P}^3 .

Theorem 2. *Every admissible torsion-free sheaf E on \mathbb{P}^3 can be obtained as the cohomology of a special monad*

$$0 \rightarrow V \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} W \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\beta} V' \otimes \mathcal{O}_{\mathbb{P}^3}(1) \rightarrow 0, \quad (2)$$

where $V = H^1(\mathbb{P}^3, E \otimes \Omega_{\mathbb{P}^3}^2(1))$, $W = H^1(\mathbb{P}^3, E \otimes \Omega_{\mathbb{P}^3}^1)$ and $V' = H^1(\mathbb{P}^3, E(-1))$.

Proof. Manin proves the case E being locally-free in [7, p. 91], using the Beilinson spectral sequence. However, the argument generalizes word by word for E being torsion-free; just note that the projection formula

$$R^i p_{1*}(p_1^* \mathcal{O}_{\mathbb{P}^3}(k) \otimes p_2^* F) = \mathcal{O}_{\mathbb{P}^3}(k) \otimes H^i(\mathbb{P}^3, F)$$

holds for every torsion-free sheaf F , where p_1 and p_2 are the natural projections of $\mathbb{P}^3 \times \mathbb{P}^3$ onto the first and second factors. \square

Clearly, the cohomology sheaf E is always coherent, but more can be said in particular situations. Note that $\alpha \in \text{Hom}(V, W) \otimes \mathcal{O}_{\mathbb{P}^1}$ and $\beta \in \text{Hom}(W, V') \otimes \mathcal{O}_{\mathbb{P}^1}$. Clearly, the surjectivity of β as a sheaf map implies that the localized map β_x is surjective for all $x \in \mathbb{P}^3$, while the injectivity of α as a sheaf map implies that the localized map α_x is injective only for generic $x \in \mathbb{P}^3$.

Theorem 3. *The cohomology E of the monad*

$$0 \rightarrow V \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{\alpha} W \otimes \mathcal{O}_{\mathbb{P}^2} \xrightarrow{\beta} V' \otimes \mathcal{O}_{\mathbb{P}^2}(1) \rightarrow 0 \quad (3)$$

is a coherent admissible sheaf with:

$$\text{rank}(E) = \dim W - \dim V - \dim V', \quad c_1(E) = \dim V' - \dim V$$

$$ch_2(E) = \frac{1}{2}(\dim V + \dim V') \quad \text{and} \quad ch_3(E) = \frac{1}{6}(\dim V - \dim V').$$

Moreover:

- E is torsion-free if and only if the localized maps α_x are injective away from a subset of codimension 2;

- E is reflexive if and only if the localized maps α_x are injective away from finitely many points;
- E is locally-free if and only if the localized maps α_x are injective for all $x \in \mathbb{P}^3$.

Proof. The kernel sheaf $\mathcal{K} = \ker \beta$ is locally-free, and one has the sequence:

$$0 \rightarrow V \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} \mathcal{K} \rightarrow E \rightarrow 0 \quad (4)$$

so E is clearly coherent. Notice also that:

$$\text{ch}(E) = \dim W - \dim V \cdot \text{ch}(\mathcal{O}_{\mathbb{P}^3}(1)) - \dim V' \cdot \text{ch}(\mathcal{O}_{\mathbb{P}^3}(1))$$

from which the calculation of the Chern classes of E follows easily.

Taking the dual of the sequence (4), we obtain:

$$0 \rightarrow E^* \rightarrow \mathcal{K}^* \xrightarrow{\alpha^*} V^* \otimes \mathcal{O}_{\mathbb{P}^3}(1) \rightarrow \text{Ext}^1(E, \mathcal{O}_{\mathbb{P}^3}) \rightarrow 0 \quad (5)$$

since \mathcal{K} is locally-free. In particular, $\text{Ext}^2(E, \mathcal{O}_{\mathbb{P}^3}) = \text{Ext}^3(E, \mathcal{O}_{\mathbb{P}^3}) = 0$ and

$$\mathcal{I} = \text{supp } \text{Ext}^1(E, \mathcal{O}_{\mathbb{P}^3}) = \{x \in \mathbb{P}^3 \mid \alpha_x \text{ is not injective} \}$$

So it is now enough to argue that \mathcal{C} is torsion-free if and only if $\dim I = 1$ and that \mathcal{C} is reflexive if and only if $\dim I = 0$; the third statement is clear.

Recall that the m^{th} -singularity set of a coherent sheaf \mathcal{F} is given by:

$$S_m(\mathcal{F}) = \{X \in \mathbb{P}^3 \mid dh(\mathcal{F}_x) \geq 3 - m\}$$

where $dh(\mathcal{F}_x)$ stands for the homological dimension of \mathcal{F}_x as an \mathcal{O}_x -module:

$$dh(\mathcal{F}_x) = d \iff \begin{cases} \text{Ext}_{\mathcal{O}_x}^d(\mathcal{F}_x, \mathcal{O}_x) \neq 0 \\ \text{Ext}_{\mathcal{O}_x}^p(\mathcal{F}_x, \mathcal{O}_x) = 0 \quad \forall p > d \end{cases}$$

In the case at hand, we have that $dh(\mathcal{F}_x) = 1$ if $X \in I$, and $dh(\mathcal{F}_x) = 0$ if $X \notin I$. Therefore $S_0(\mathcal{C}) = S_1(\mathcal{C}) = \emptyset$, while $S_2(\mathcal{C}) = I$. It follows that [9, Proposition 1.20] :

- if $\dim I = 1$, then $\dim S_m(\mathcal{C}) \leq m - 1$ for all $m < 3$, hence \mathcal{C} is a locally 1st-syzygy sheaf;
- if $\dim I = 0$, then $\dim S_m(\mathcal{C}) \leq m - 2$ for all $m < 3$, hence \mathcal{C} is a locally 2nd-syzygy sheaf.

The desired statements follow from the observation that \mathcal{C} is torsion-free if and only if it is a locally 1st-syzygy sheaf, while \mathcal{C} is reflexive if and only if it is a locally 2nd-syzygy sheaf [8, p. 148-149]. \square

As part of the proof above, it is worth emphasizing that if E is admissible then $\text{Ext}^2(E, \mathcal{O}_{\mathbb{P}^3}) = \text{Ext}^3(E, \mathcal{O}_{\mathbb{P}^3}) = 0$.

It follows from Theorems 2 and 3 that there exists a (set theoretical) 1-1 correspondence between special monads and admissible sheaves. As shown by Manin and Drinfeld (see [7]), such correspondence is categorical.

Theorem 4. *The functor that associates a special monad on \mathbb{P}^3 to its cohomology sheaf defines an equivalence between the categories of special monads and admissible sheaves.*

We complete this section with an important fact:

Proposition 5. *If E is an admissible sheaf, then $H^0(\mathbb{P}^2, E^*(k)) = 0$ for all $k \leq -1$.*

Proof. E is the cohomology of the monad (2); setting $V = H^1(\mathbb{P}^2, E(-2))$, $W = H^1(\mathbb{P}^2, E \otimes \Omega_{\mathbb{P}^2}^1(-1))$ and $V' = H^1(\mathbb{P}^2, E(-1))$, one had the sequences

$$0 \rightarrow \mathcal{K}(k) \rightarrow W \otimes \mathcal{O}_{\mathbb{P}^2}(k) \rightarrow V' \otimes \mathcal{O}_{\mathbb{P}^2}(k+1) \rightarrow 0 \quad \text{and}$$

$$0 \rightarrow V \otimes \mathcal{O}_{\mathbb{P}^2}(k-1) \rightarrow \mathcal{K}(k) \rightarrow E(k) \rightarrow 0 .$$

where $\mathcal{K} = \ker\{W \otimes \mathcal{O}_{\mathbb{P}^2} \rightarrow V' \otimes \mathcal{O}_{\mathbb{P}^2}\}$ is a locally-free sheaf. It follows from the first sequence that:

$$H^0(\mathbb{P}^2, \mathcal{K}(k)) = 0 \quad \forall k \leq -1, \quad H^2(\mathbb{P}^2, \mathcal{K}(k)) = 0 \quad \forall k \geq -2$$

$$\text{and } H^0(\mathbb{P}^2, \mathcal{K}^*(k)) = 0 \quad \forall k \leq -1, \quad \text{by Serre duality.}$$

The proposition then follows easily from the dual of the second sequence. \square

2 Examples of admissible sheaves

Let us now study various examples of admissible sheaves on \mathbb{P}^3 . Theorem 1 implies that there are admissible coherent sheaves in rank 0 and 1, but there can be no admissible sheaves with zero first Chern class in these ranks, apart from the trivial ones.

Examples of admissible sheaves with vanishing first Chern class start in rank 2. The basic one is an admissible torsion-free sheaf E which is not locally-free; it arises as the cohomology E of the monad:

$$\mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^3}^{\oplus 4} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(1) \quad (6)$$

$$\alpha = \begin{pmatrix} x \\ y \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \beta = (-y \ x \ z \ w)$$

It is easy to see that β is surjective for all $[x : y : z : w] \in \mathbb{P}^3$, while α is injective provided $x, y \neq 0$. It then follows from Theorem 3 that E is torsion-free, but not locally-free. In particular, the singularity set of E (i.e. the support of E^{**}/E) consists of the line $\{x = y = 0\} \subset \mathbb{P}^3$. Note also that $c_2(E) = 1$ and $c_1(E) = c_3(E) = 0$.

Reflexive sheaves on \mathbb{P}^3 have been extensively studied in a series of papers by Hartshorne [6], among other authors. In particular, it was shown that a rank 2 reflexive sheaf \mathcal{F} on \mathbb{P}^3 is locally-free if and only if $c_3(\mathcal{F}) = 0$. Therefore, we conclude:

Proposition 6. (Hartshorne [6]) *There are no rank 2 admissible sheaves on \mathbb{P}^3 which are reflexive but not locally-free.*

The situation for higher rank is quite different, though, and it is easy to construct a rank 3 admissible sheaf which is reflexive but not locally-free. Setting $w = 5$ and $v = v' = 1$, consider the monad:

$$\mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^3}^{\oplus 5} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(1) \quad (7)$$

$$\alpha = \begin{pmatrix} x \\ y \\ 0 \\ 0 \\ z \end{pmatrix} \quad \text{and} \quad \beta = (-y \ x \ z \ w \ 0)$$

Again, it is easy to see that β is surjective for all $[x : y : z : w] \in \mathbb{P}^3$, while α is injective provided $x, y, z \neq 0$. It then follows from Theorem 3 that EE is reflexive, but not locally-free; its singularity set is just the point $[0 : 0 : 0 : 1] \in \mathbb{P}^3$. Note also that $c_2(E) = 1$ and $c_1(E) = c_3(E) = 0$.

Finally, we give an example of a rank 2 admissible locally-free sheaf. Setting $w = 4$ and $v = v' = 1$, consider the monad:

$$\mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^3}^{\oplus 4} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(1) \quad (8)$$

$$\alpha = \begin{pmatrix} x \\ y \\ -w \\ z \end{pmatrix} \quad \text{and} \quad \beta = (-y \ x \ z \ w)$$

It is easy to see that β is surjective and α is injective for all $[x : y : z : w] \in \mathbb{P}^3$, so E is indeed locally-free; note that $c_2(E) = 1$ and $c_1(E) = c_3(E) = 0$.

With these simple examples in low rank, we can produce high rank admissible sheaves using the following:

Proposition 7. *If F' and F'' are coherent admissible sheaves, its extention E :*

$$0 \rightarrow F' \rightarrow E \rightarrow F'' \rightarrow 0$$

is also admissible.

The proof is an easy consequence of the associated long exact sequence in cohomology, and it is left to the reader. As a consequence of Serre duality, we have:

Proposition 8. *If E is a locally-free admissible sheaf, then E^* is also admissible.*

3 Semistability of torsion-free admissible sheaves

Recall that a torsion-free sheaf E on \mathbb{P}^3 is said to be *semistable* if for every coherent subsheaf $0 \neq F \hookrightarrow E$ we have

$$\mu(F) = \frac{c_1(F)}{\text{rk}(F)} \leq \frac{c_1(E)}{\text{rk}(E)} = \mu(E) .$$

Furthermore, if for every coherent subsheaf $0 \neq F \hookrightarrow E$ with $0 < \text{rk}(F) < \text{rk}(E)$ we have

$$\frac{c_1(F)}{\text{rk}(F)} < \frac{c_1(E)}{\text{rk}(E)} ,$$

then E is said to be *stable*. It is also important to remember that:

- E is (semi)stable if and only if E^* is;
- E is (semi)stable if and only if $\mu(F) < \mu(E)$ ($\mu(F) \leq \mu(E)$) for all coherent subsheaves $F \hookrightarrow E$ whose quotient E/F is torsion-free;
- E is (semi)stable if and only if $\mu(Q) > \mu(E)$ ($\mu(Q) \geq \mu(E)$) for all torsion-free quotients $E \rightarrow Q \rightarrow 0$ with $0 < \text{rk}(Q) < \text{rk}(E)$.

Furthermore, if E is locally-free, it is enough to test the locally-free subsheaves $F \hookrightarrow E$ with $0 < \text{rk}(F) < \text{rk}(E)$ to conclude that E is stable.

The goal of this section is to compare the semistability and admissibility conditions. We provide to positive results for admissible sheaves of rank 2 and 3.

Theorem 9. *Let E be a semistable torsion-free sheaf with $c_1(E) = 0$. E is admissible if and only if $H^1(\mathbb{P}^3, E(-2)) = H^2(\mathbb{P}^3, E(-2)) = 0$. Furthermore, an admissible torsion-free sheaf is stable if and only if $H^0(\mathbb{P}^3, E) = 0$.*

In other words, if E is a semistable torsion-free sheaf with $c_1(E) = 0$ and $H^1(\mathbb{P}^3, E(-2)) = H^2(\mathbb{P}^3, E(-2)) = 0$, then E is admissible.

Proof. Semistability implies immediately that $H^0(\mathbb{P}^3, E(k)) = 0$, for all $k \leq 1$. If E is locally-free, then by Serre duality we have $H^3(\mathbb{P}^3, E(k)) = 0$ for all $k \geq -3$, since E^* is also semistable. If E is torsion-free, we can use the semistability of E^{**} and the sequence

$$0 \rightarrow E \rightarrow E^{**} \rightarrow Q \rightarrow 0 \quad , \quad Q = E^{**}/E$$

to conclude that $H^3(\mathbb{P}^3, E(k)) = H^3(\mathbb{P}^3, E^{**}(k)) = 0$, since Q is supported in dimension less or equal to 1.

Now we assume that $H^1(\mathbb{P}^3, E(-2)) = H^2(\mathbb{P}^3, E(-2)) = 0$, and let \wp be a plane in \mathbb{P}^3 . From the sequence:

$$0 \rightarrow E(k-1) \rightarrow E(k) \rightarrow E|_{\wp}(k) \rightarrow 0 \quad (9)$$

we conclude that $H^0(\wp, E|_{\wp}(-1)) = H^2(\wp, E|_{\wp}(-2)) = 0$.

Claim. *If V is a torsion-free sheaf on \mathbb{P}^2 with $H^0(\mathbb{P}^2, V(-1)) = H^2(\mathbb{P}^2, V(-2)) = 0$, then $H^0(\mathbb{P}^2, V(k)) = 0$ for $k \leq -1$ and $H^2(\mathbb{P}^2, V(k)) = 0$ for $k \geq -2$.*

Proof of the claim: For any line $\ell \subset \mathbb{P}^2$, we have the sequence

$$0 \rightarrow V(k-1) \rightarrow V(k) \rightarrow V|_{\ell}(k) \rightarrow 0 \quad ,$$

so that

$$0 \rightarrow H^0(\mathbb{P}^2, V(k-1)) \rightarrow H^0(\mathbb{P}^2, V(k)) \quad \text{and} \quad H^2(\mathbb{P}^2, V(k-1)) \rightarrow H^2(\mathbb{P}^2, V(k)) \rightarrow 0 \quad .$$

The claim follows easily by induction.

Returning to (9), we also have:

$$H^0(\wp, E|_{\wp}(k)) \rightarrow H^1(\mathbb{P}^3, E(k-1)) \rightarrow H^1(\mathbb{P}^3, E(k))$$

so, for $k \leq -1$, if $H^1(\mathbb{P}^3, E(k)) = 0$, then also $H^1(\mathbb{P}^3, E(k-1)) = 0$. Thus by induction we conclude that $H^1(\mathbb{P}^3, E(k)) = 0$ for all $k \leq -2$.

Similarly, we have:

$$H^2(\mathbb{P}^3, E(k-1)) \rightarrow H^2(\mathbb{P}^3, E(k)) \rightarrow H^2(\wp, E|_{\wp}(k))$$

and again by induction we conclude that $H^2(\mathbb{P}^3, E(k)) = 0$ for all $k \geq -2$, as desired. \square

The converse statement seems to depend on the rank, as we will see in the two results below.

Theorem 10. *Every rank 2 admissible torsion-free sheaf E with $c_1(E) = 0$ is semistable. Moreover, if $H^0(\mathbb{P}^3, E^*) = 0$, then E is stable.*

Proof. First, assume that L is a rank 2 reflexive sheaf with $c_1(L) = 0$ and $H^0(L(k)) = 0$ for all $k \leq -1$; we show that L is semistable. Indeed, let $F \hookrightarrow L$ be a torsion-free subsheaf of rank 1, with torsion-free quotient $Q = L/F$. By Lemma 1.1.16 in [8, p. 158], it follows that F is also reflexive; but every rank 1 reflexive sheaf is locally-free, thus $F = \mathcal{O}_{\mathbb{P}^3}(d)$. Any map $F \rightarrow L$ yields a section in $H^0(\mathbb{P}^3, L(-d))$, $c_1(F) = d \leq 0$ and E is semistable, being stable if $H^0(\mathbb{P}^3, E^*) = 0$.

Now if E is a rank 2 admissible torsion-free sheaf with $c_1(E) = 0$, then $L = E^*$ is a rank 2 reflexive sheaf with $c_1(L) = 0$ and $H^0(L(k)) = 0$ for all $k \leq -1$, by Proposition 5 and since the dual of any coherent sheaf is always reflexive. Thus E^* is semistable, so E is as well. Clearly, E is stable if $H^0(\mathbb{P}^3, E^*) = 0$, as desired. \square

A similar result for rank 3 sheaves requires a stronger hypothesis: reflexivity, rather than torsion-freeness.

Theorem 11. *Every rank 3 admissible reflexive sheaf E with $c_1(E) = 0$ is semistable. Moreover, if $H^0(\mathbb{P}^3, E) = H^0(\mathbb{P}^3, E^*) = 0$, then E is stable.*

Proof. In fact, one can show that every rank 3 reflexive sheaf with $c_1(E) = 0$ and $H^0(E(k)) = H^0(E^*(k)) = 0$ is semistable. The desired theorem follows easily from this fact.

Indeed, let $F \hookrightarrow E$ be a torsion-free subsheaf, with torsion-free quotient $Q = E/F$, so that $c_1(F) = -c_1(Q)$. As in the proof of Theorem 10, it follows that F is reflexive. There are two possibilities:

- (i) rank $F = 1$. In this case, F is locally-free, so a map $F \rightarrow E$ yields a section in $H^0(\mathbb{P}^3, E(-d))$, where $d = c_1(F)$. Hence $c_1(F) \leq 0$.
- (ii) rank $F = 2$, so rank $Q = 1$. Now Q^* is a reflexive (hence locally-free) subsheaf of E^* , which gives a section in $H^0(\mathbb{P}^3, E^*(-d))$, where $d = c_1(Q^*) = c_1(F)$. Hence $c_1(F) \leq 0$.

It follows that E is semistable, being stable if $H^0(\mathbb{P}^3, E) = H^0(\mathbb{P}^3, E^*) = 0$. \square

Together with Theorem 9, we conclude that:

- a rank 2 torsion-free sheaf on \mathbb{P}^3 with $c_1(E) = 0$ is admissible if and only if it is semistable and $H^1(\mathbb{P}^3, E(-2)) = H^2(\mathbb{P}^3, E(-2)) = 0$;
- a rank 3 reflexive sheaf on \mathbb{P}^3 with $c_1(E) = 0$ is admissible if and only if it is semistable and $H^1(\mathbb{P}^3, E(-2)) = H^2(\mathbb{P}^3, E(-2)) = 0$.

4 Trivial splitting type

Since every locally-free sheaf on a projective line splits as a sum of line bundles, one can study sheaves on projective spaces by looking into the behavior of restriction to a line [8].

Definition. A torsion-free sheaf E on \mathbb{P}^3 is said to be of *trivial splitting type* if there is a line $\ell \subset \mathbb{P}^3$ such that $E|_\ell$ is the trivial locally-free sheaf, i.e. $E|_\ell \simeq \mathcal{O}_\ell^{\oplus \text{rk} E}$.

A sheaf of trivial splitting type necessarily has vanishing first Chern class. Note that, by semicontinuity, if E is of trivial splitting type then $E|_\ell$ is trivial for a generic line in \mathbb{P}^3 . Torsion-free sheaves of trivial splitting type were completely classified in [4], and they were shown to be closely related with a complex version of the celebrated Atiyah-Drinfeld-Hitchin-Manin matrix equations.

Furthermore, every torsion-free sheaf of trivial splitting type is semistable; indeed, assume that E has rank r , and let $F \hookrightarrow E$ be a coherent subsheaf of rank s , with torsion-free quotient E/F . Then on a generic line $\ell \subset \mathbb{P}^3$ we have:

$$F_\ell = \mathcal{O}_\ell(a_1) \oplus \cdots \oplus \mathcal{O}_\ell(a_s) \hookrightarrow E|_\ell \simeq \mathcal{O}_\ell^{\oplus r} \quad ,$$

where $c_1(F) = a_1 + \cdots + a_s$. It follows that $c_1(F) \leq 0$, since we must have $a_k \leq 0$, $k = 1, \dots, s$.

Theorem 12. *Let E be a torsion-free sheaf of trivial splitting type. E is admissible if and only if $H^1(\mathbb{P}^3, E(-2)) = H^2(\mathbb{P}^3, E(-2)) = 0$.*

Of course, this is an easy consequence of Theorem 9 and the observation above, but here is a direct proof.

Proof. Let E be an admissible torsion-free sheaf. Without loss of generality, we can assume that $E|_{\ell_\infty}$ is trivial for $\ell_\infty = \{z = w = 0\}$. Let \wp be a plane containing ℓ_∞ , e.g. $\wp = \{z = 0\}$. Then $E|_\wp$ is a torsion-free sheaf on \wp which is trivial at ℓ_∞ . From the proof of Theorem 9 we know that:

$$H^0(\wp, E|_\wp(k)) = 0 \quad \forall k \leq -1 \quad , \quad H^2(\wp, E|_\wp(k)) = 0 \quad \forall k \geq -2 \quad (10)$$

Now consider the sheaf sequence:

$$0 \rightarrow E(k-1) \xrightarrow{\cdot z} E(k) \rightarrow E|_\wp(k) \rightarrow 0 \quad (11)$$

Using (10), we conclude that:

$$H^3(\mathbb{P}^3, E(k)) = H^3(\mathbb{P}^3, E(k-1)) \quad \forall k \geq -2$$

But, by Serre's vanishing theorem, $H^3(\mathbb{P}^3, E(N)) = 0$ for sufficiently large N , thus $H^3(\mathbb{P}^3, E(k)) = 0$ for all $k \geq -3$.

Similarly, we have:

$$H^0(\mathbb{P}^3, E(k-1)) = H^0(\mathbb{P}^3, E(k)) \quad \forall k \leq -1$$

Since $E \hookrightarrow E^{**}$, we have via Serre duality:

$$H^0(\mathbb{P}^3, E(k)) \hookrightarrow H^0(\mathbb{P}^3, E^{**}(k)) = H^3(\mathbb{P}^3, E^{***}(-k-4))^* .$$

Thus, again by Serre's vanishing theorem, $H^0(\mathbb{P}^3, E(-N)) = 0$ for sufficiently large N , so that $H^0(\mathbb{P}^3, E(k)) = 0$ for all $k \leq -1$.

We also have that:

$$0 \rightarrow H^1(\mathbb{P}^3, E(k-1)) \rightarrow H^1(\mathbb{P}^3, E(k)) \quad \forall k \leq -1$$

hence $H^1(\mathbb{P}^3, E(-2)) = 0$ implies that $H^1(\mathbb{P}^3, E(k)) = 0$ for all $k \leq -2$. Furthermore,

$$H^2(\mathbb{P}^3, E(k-1)) \rightarrow H^2(\mathbb{P}^3, E(k)) \rightarrow 0 \quad \forall k \geq -2$$

forces $H^2(\mathbb{P}^3, E(k)) = 0$ for all $k \geq -2$ once $H^2(\mathbb{P}^3, E(-2)) = 0$. \square

As in the previous section, the converse statement seems to depend on the rank. The generic splitting type of a semistable locally-free sheaf with vanishing first Chern class is determined by Theorem 2.1.4 in [8, p. 205-206]. In particular, it follows that every semistable rank 2 locally-free sheaf is of trivial splitting type. Thus, from Theorem 10, we conclude:

Theorem 13. *Every rank 2 admissible locally-free sheaf E with $c_1(E) = 0$ is of trivial splitting type.*

To explore a few easy consequences of the classical theory of locally-free sheaves on complex projective spaces, let \mathbb{G} denote the Grassmannian of lines in \mathbb{P}^3 .

Definition. *Let E be a locally-free sheaf of trivial splitting type. The set*

$$\mathcal{J}_E = \{\ell \in \mathbb{G} \mid E_\ell \text{ is not trivial}\}$$

is called the set of jumping lines of E ; it is always a closed subvariety of \mathbb{G} . Moreover, E is said to be uniform if \mathcal{J}_E is empty, i.e. if E_ℓ is independent of $\ell \in \mathbb{G}$.

Theorem 14. *Every rank 2 uniform, admissible locally-free sheaf E with $c_1(E) = 0$ is trivial.*

Proof. By Theorem 13, E is of trivial splitting type. Since E is uniform, E_ℓ must be trivial for all $\ell \in \mathbb{G}$. It then follows from Theorem 3.2.1 in [8, p. 51] that E is trivial. \square

Our last result, regarding the set of jumping lines, follows from Theorem 2.2.3 in [8, p. 228].

Theorem 15. *If E is an rank 2 admissible locally-free sheaf with $c_1(E) = 0$, then its set of jumping lines \mathcal{J}_E is a subvariety of pure codimension 1 in \mathbb{G} .*

5 Open problems

The results proved in this paper point to a number of quite interesting questions and possible generalizations. First of all, we expect that if E is a properly torsion-free or properly reflexive admissible sheaf, then its dual E^* is not admissible, but we have not been able to construct any examples.

We would also like to see whether the results in Section 3 can be generalized to higher rank. It seems too much to expect every admissible sheaf to be semistable; but the correspondence between instantons and locally-free admissible sheaves makes the statement "every admissible locally-free sheaf is semistable" an attractive conjecture. On the other hand, is Theorem 11 optimal, i.e. is there a rank 3 torsion-free admissible sheaf which is not semistable?

It would also be interesting to study the connection between admissibility and Gieseker stability. Since every Gieseker semistable sheaf on a projective space is also semistable, we conclude from Theorem 9 that every Gieseker semistable torsion-free sheaf is admissible; one would like to determine to what extent the converse is also true.

Theorems 13 and 14 point to interesting properties of higher rank admissible sheaves: is every admissible locally-free sheaf with vanishing first Chern class of trivial splitting type? Is every uniform, admissible locally-free sheaf with vanishing first Chern class trivial?

We've also seen that if E is an admissible torsion-free and \wp is a plane in \mathbb{P}^3 , then the restriction $E|_{\wp}$ satisfies the following cohomological condition:

$$H^0(\wp, E|_{\wp}(k)) = 0 \quad \forall k \leq -1 \quad , \quad H^2(\wp, E|_{\wp}(k)) = 0 \quad \forall k \geq -2 \quad .$$

A sheaf on \mathbb{P}^2 satisfying the above conditions are called *instanton sheaves*, and are very interesting on their own right, also being closely related to instantons. The analysis of how the instanton condition compares with semistability and trivial splitting type is work in progress [5], but many of the results proved here have their analogs for instanton sheaves in \mathbb{P}^2 .

In particular, it is shown in [5] that every instanton sheaf is the cohomology of a special monad, and that every rank 2 torsion-free instanton sheaf is semistable. We can then conjecture that if a (rank 2) torsion-free sheaf E on \mathbb{P}^k is the cohomology of a special monad, then E is semistable; this is true for $k = 2, 3$.

References

- [1] Atiyah, M., Drinfeld, V., Hitchin, N., Manin, Yu.: Construction of instantons. Phys. Lett. **65A**, 185-187 (1978)
- [2] Coandă, I., Tikhomirov, A., Trautmann, G.: Irreducibility and smoothness of the moduli space of mathematical 5-instantons over \mathbb{P}^3 . Int. J. Math. **14**, 1-45 (2003)
- [3] Floystad, G.: Monads on projective spaces. Comm. Algebra **28**, 5503-5516 (2000)
- [4] Frenkel, I., Jardim, M.: Complex ADHM equations, sheaves on \mathbb{P}^3 and quantum instantons. Preprint math.RT/0408027.

- [5] Jardim, M.: Instanton sheaves on \mathbb{P}^2 . In preparation.
- [6] Hartshorne, R.: Stable reflexive sheaves I, II, III. Math. Ann. **254**, 121-176 (1980) Invent. Math. **66**, 165-190 (1982) Math. Ann. **279**, 517–534 (1988)
- [7] Manin, Yu.: *Gauge field theory and complex geometry*. Berlin: Springer-Verlag, 1997 (second edition)
- [8] Okonek, O., Schneider, M., Spindler, H.: *Vector bundles on complex projective spaces*. Boston: Birkhauser (1980)
- [9] Siu, Y.-T., Trautmann, G.: *Gap-sheaves and extension of coherent analytic subsheaves*. Lec. Notes Math, **172**. Berlin: Springer-Verlag (1971)
- [10] Ward, R., Wells, R.: *Twistor geometry and field theory*. Cambridge: Cambridge University Press (1990)